

# Low $x$ Physics and Perturbative Regge Asymptotics<sup>1</sup>

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## Abstract

Deep-inelastic scattering at small  $x$  is a new field of QCD phenomenology which HERA experiments started to explore. The data show the feature expected by the perturbative Regge asymptotics. We describe the parton picture of structure function evolution both for increasing  $Q^2$  and for decreasing  $x$  and discuss the leading logarithmic approximation of QCD on which this picture is based. The steep rise of the parton density towards smaller  $x$  gets saturated by parton recombination. An essential improvement of the leading logarithmic approximation has to be worked out in order to satisfy the unitarity conditions and in this way to describe the parton recombination in perturbative QCD. We review some ideas of the perturbative Regge asymptotics and some recent theoretical results about multi-reggeon exchange in QCD.

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# 1 Deep-inelastic scattering at small $x$

The parton distribution obtained from deep-inelastic structure functions enter the expression for all hard processes. At high energies the small- $x$  region of the distributions gives the dominant contribution. The measurements at HERA [1] [2] of  $F_2(x, Q^2)$  at values of  $x$  down to  $10^{-4}$  reduce essentially the uncertainties in this region. Moreover, HERA provides the first experimental information about an unexplored field of QCD phenomenology.

At small  $x$  the virtual Compton amplitude, the forward imaginary part of which determines the structure functions, is in the Regge regime. Indeed, we have for the c.m. energy squared of the virtual photon and the proton

$$s \simeq \frac{Q^2}{x} \gg Q^2 \gg m_p^2. \quad (1.1)$$

It is a peculiar Regge limit because of  $Q^2$  being much larger than the hadronic scale (represented by the proton mass  $m_p$ ). This provides the advantage that an essential part of the deep-inelastic interaction can be calculated perturbatively also at small  $x$ .

The perturbative Regge asymptotics in non-abelian gauge theories has been a topic of theoretical investigation already 20 years ago. In the leading logarithmic approximation the asymptotics for vacuum quantum number exchange is determined [3] [4] by a branch point singularity at the angular momentum  $1 + \omega_0$ ,  $\omega_0 = \frac{g^2 N}{2\pi^2} 2 \ln 2$ . The perturbative contributions lead to a power-like behaviour of the structure function at small  $x$ ,

$$F_2(x, Q^2) \sim \left(\frac{1}{x}\right)^{\omega_0}. \quad (1.2)$$

Identifying  $\frac{g^2}{4\pi}$  with  $\alpha_S(10\text{GeV}^2)$  and choosing  $N = 3$  one estimates  $\omega_0 = 0.5$ . This is a much steeper increase towards smaller  $x$  compared to the expectations from hadron Regge phenomenology (pomeron intercept at 1.08) or to the asymptotics induced by the usual  $Q^2$ -evolution of structure functions ( $\sim \exp(B(-\ln x)^{1/2})$ , called  $r$ -evolution below). The data give evidence that this increase, Eq. (2), really takes place. The seemingly smaller power in the data at lower  $Q^2$  is a subject of a more detailed discussion.

The perturbative Regge asymptotics is expected to show up not only in the  $x$ -dependence but also in certain features of the hadronic final state, which is a subject of experimental [5] [6] and theoretical [7] [8] [9] investigations.

## 2 The parton picture

Let us draw the parton picture of the structure function evolution before we discuss some details about how it emerges from QCD.

The parton hit by the virtual photon can be considered as emerging from an evolution starting from the hadron constituents and proceeding by parton radiation. At each step  $\ell$  of the radiation process the transverse momentum  $\kappa_\ell$  and the longitudinal momentum fraction  $x_\ell$  of the parton change.  $x$  is the value of  $x_\ell$  at the final step ( $\ell = n$ ) of the evolution and  $Q$  is the upper bound for  $|\kappa|$ . If  $x$  is small then there is room for this

process to evolve both in  $r = \ln(|\kappa^2|/m_p^2)$  as well as in  $y = -\ln x$ . The  $y - r$  plot Fig. 1 serves as an useful illustration [10]. The features of the evolution in  $r$  are well known. At higher steps in this evolution the partons appear in higher multiplicity and at higher resolution (a reasonable measure of which is just  $r$ ). The new partons cover a negligible fraction of the transverse space occupied by the parent parton.

The evolution in  $y$  proceeds without essential changes in the resolution. The new partons appear with about the same size as the parent parton. According to Eq.(2) the multiplicity grows exponentially in  $y$ . The partons tend to cover the space inside the proton. The rapid growth of the multiplicity in the first stage of the  $y$ -evolution becomes saturated by recombination effects before the partons come too close to each other. The region where the recombination is essential is bounded by the critical line in the  $y - r$  plot Fig.1.

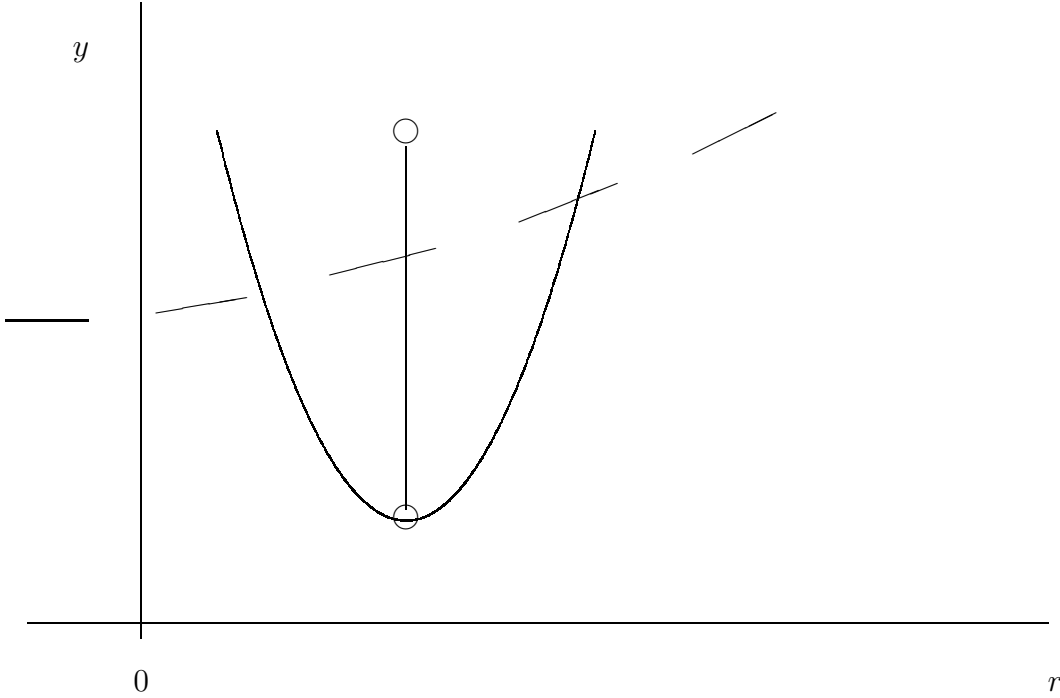


Fig.1: The evolution of the parton density illustrated in the  $y - r$  plot. Left to the  $y$  axis is the non-perturbative region. The dashed line indicates the critical line and the parabola shows the  $r$ -range broadening due to diffusion in the  $y$ -evolution.

During the evolution in  $y$  the value of  $r$  undergoes a random walk and may run into the non-perturbative region ( $r < 0$ ). This means in particular that the value of the parton density at given  $r$  and  $y$  becomes influenced by non-perturbative contributions if  $y$  becomes large enough even if  $r$  is far from the non-perturbative region.

In both the  $r$ -evolution and the  $y$ -evolution the process starts with the proton constituents inside the non-perturbative region. Measuring a jet with  $\kappa_j, x_j$  in the hadronic final state one detects an intermediate state of the evolution in the perturbative region

(provided  $\kappa_j$  is not small). In the case  $x_j \gg x$ ,  $|\kappa_j^2| \sim Q^2$  this measurement selects a nearly ideal perturbative  $y$ -evolution, all features of which are calculable [7].

### 3 The leading logarithmic approximation

In certain kinematical regions large perturbative correction in each order arise from the integration over  $\kappa$  or  $x$  of (real or virtual) partons, if the range in  $\kappa$  is large or if the range in  $x$  extends to small values and if the whole range contributes uniformly. In this case the corresponding integrals are approximately of logarithmic form.

In the leading  $\ln \kappa^2$  approximation one picks up the logarithmic contributions from the transverse momentum integral of each loop. Graphs with non-logarithmic loops are neglected. Applied to the structure functions at  $r > y$  this is the appropriate approximation which can be improved systematically. The leading contribution can be represented as a sum of ladder graphs (in an axial gauge, with propagator and vertex corrections included). There is a strong ordering of the transverse momenta in the loops which leads in particular to trivial transverse momentum integrals. Summing these graphs leads to the known GLAP equation [11]. The iterative structure of the ladders implies in particular the following form of the structure functions, exhibiting their universality.

$$F_2(x, Q^2) = \frac{d\sigma^{\gamma*j}}{d\hat{t}}|_{i=0} \otimes D_{ji}(\frac{x_1}{x_2}, Q^2, Q_0^2) \otimes F_i^{(0)}(x_2, Q_0^2) \quad (3.3)$$

$\otimes$  abbreviates a convolution integral in the longitudinal momentum fraction. A summation over the parton types  $i, j$  is understood. The first factor represents the forward cross section for the virtual photon - parton scattering. Replacing it by other hard scattering cross sections allows to relate the structure functions to distributions in those processes. The second factor represents the  $r$ -evolution, the solution of the GLAP equation with the initial distribution  $\delta_{ij}\delta(\frac{x_1}{x_2} - 1)$  at  $Q_0^2$ .

Since the first two factors are calculated from QCD all experimental information about structure functions and hard processes can be reduced to the distributions  $F_i^{(0)}(x, Q_0^2)$  of partons in the hadron at  $Q_0^2$ .

In the leading  $\ln x$  approximation the integrations over  $x$  and  $\kappa$  interchange their roles compared to the above case. For structure functions it is the appropriate approximation at  $y > r$ . Again the leading contribution can be represented as a sum of ladder graphs, Fig. 2.

Now the lines in the  $t$ -channel direction represent reggeized gluons and the interaction is determined by effective vertices. A strong ordering holds now in the longitudinal momenta. More precisely, the loop momenta  $k_\ell$  obey the conditions of multi-Regge kinematics which can be written using the Sudakov decomposition,

$$k = \sqrt{\frac{2}{s}} (k_- q' + k_+ p) + \kappa, \quad q' = q - xp, \quad (3.4)$$

( $q$  and  $p$  are the 4-momenta of the photon and the proton), in the following form

$$k_{+n} \ll \dots \ll k_{+1}, \quad k_{-n} \gg \dots \gg k_{-1},$$

$$k_{+\ell}k_{-\ell} \ll |\kappa_\ell^2|, \quad s_\ell \equiv k_{+\ell-1}k_{-\ell+1} \gg |\kappa_\ell^2|, \quad \prod_{\ell=1}^n s_\ell = s \prod_{\ell=2}^n |\kappa_\ell - \kappa_{\ell-1}|^2. \quad (3.5)$$

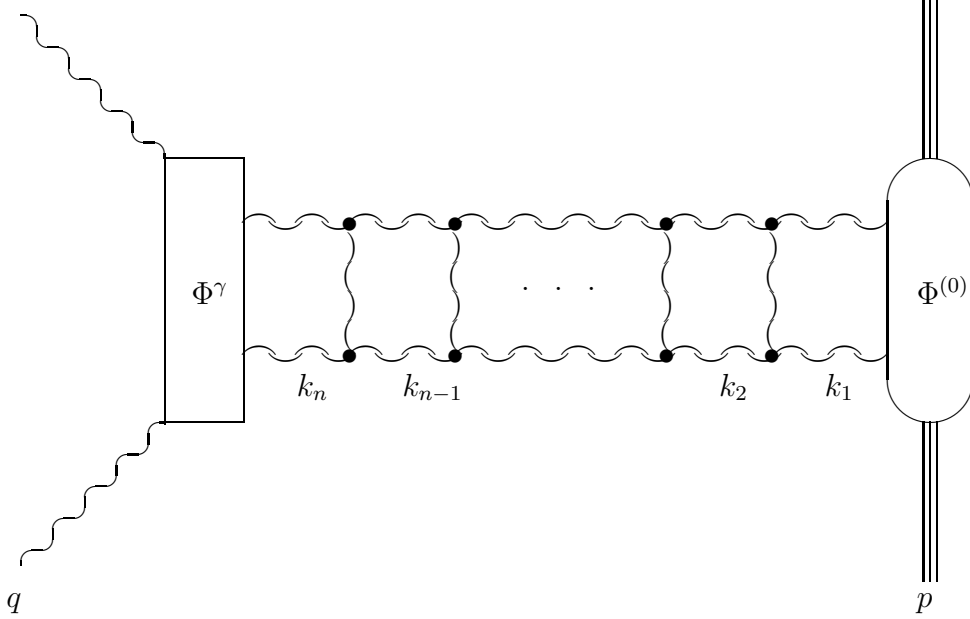


Fig. 2: Contribution to the structure function in the leading  $\ln x$  approximation.

The longitudinal momentum integrals become trivial. The iterative structure of the ladders leads to the following form of the resulting structure functions,

$$F_2(x, Q^2) = \Phi^{\gamma*}(\frac{x}{x_1}, Q, \kappa) \otimes \tilde{f}(\frac{x_1}{x_2}, \kappa, \kappa_0) \otimes \Phi^{(0)}(x_2, \kappa_0). \quad (3.6)$$

Unlike in Eq. (3) here  $\otimes$  denotes a transverse momentum integration besides of a trivial  $x$ -integral. The universality of the structure functions holds in the  $y$ -evolution with this modification, the phenomenological relevance of which has been discussed recently [12]. Here the first factor represents the impact factor coupling (via a quark loop) the virtual photon to the gluon ladder. The second factor describes the  $y$ -evolution. Because the first two factors are calculated in QCD the non-trivial information is reduced to the third factor, the hadron impact factor.

The  $y$ -evolution is calculated as the solution of the BFKL equation [3] [4], a simple form of which (at vanishing momentum transfer) is given by

$$\frac{\partial}{\partial y} \tilde{f}(y, \kappa, \kappa_0) = \frac{g^2 N}{(2\pi)^3} \int \frac{d^2 \kappa'}{|\kappa - \kappa'|^2} \frac{2|\kappa^2|}{|\kappa'^2|} \left( \tilde{f}(y, \kappa', \kappa_0) - \frac{1}{2} \tilde{f}(y, \kappa, \kappa_0) \right). \quad (3.7)$$

The asymptotics of the solution at large  $y$ ,

$$\tilde{f}(y, \kappa, \kappa_0) \sim \frac{|\kappa|}{(2\pi D^2 y)^{1/2}} e^{\omega_0 y} \exp\left(\frac{-\ln^2(|\kappa^2|/|\kappa_0^2|)}{2D^2 y}\right),$$

$$\omega_0 = \frac{g^2 N}{2\pi^2} 2 \ln 2, \quad D^2 = \frac{g^2 N}{\pi^2} 7 \zeta(3), \quad (3.8)$$

exhibits the power-like increase in  $\frac{1}{x}$  and the diffusion in  $r$  mentioned above.

Structure function parametrizations [13] [14] [15] rely on Eq. (3) and look for the optimal parton distributions  $F_i^{(0)}(x, Q_0^2)$  describing all data (Fig. 3). The low- $x$  data are well described by assuming the behaviour Eq. (2). There is a parametrization approach [15] which manages to describe the rise at small  $x$  without assuming a rising initial parton distribution like Eq. (2). The point is that the small  $x$  asymptotics induced by the GLAP equation  $\sim \exp(B(-\ln x)^{1/2})$  can be close to the rise originating from an input distribution like Eq. (2) at not too small  $x$  by choosing the starting point  $Q_0^2$  correspondingly (actually  $B^2 = \frac{4N\alpha_S}{\pi} \ln \frac{Q^2}{Q_0^2}$ ). However a choice of  $Q_0$  deep in the non-perturbative region which is necessary to describe the data in that way cannot be justified.

In the small- $x$  region a complimentary approach to the parametrization can be considered, relying on the  $y$ -evolution Eqs. (6), (7) and parametrizing the hadron impact factor  $\Phi^{(0)}(x_0, \kappa)$  at some initial  $y_0 = \ln \frac{1}{x_0}$  as a function of  $\kappa$  (Fig. 3). The behaviour of this impact factor at large  $\kappa^2$  has to be compatible with the GLAP equation. Some steps along this line have been done [16].

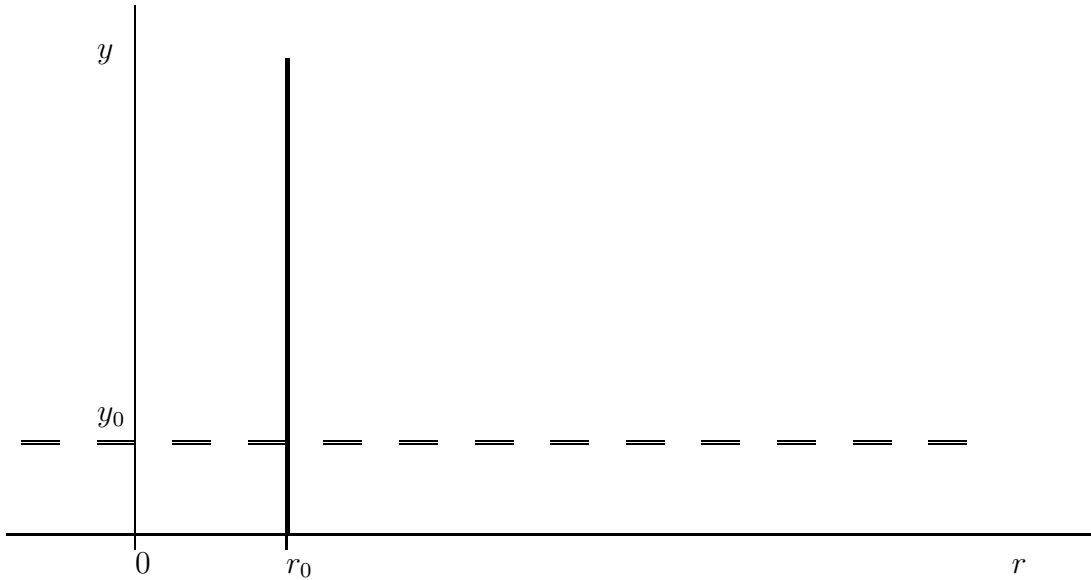


Fig. 3: The parametrization of parton densities illustrated in the  $y - r$  plot. The initial values of the hadron structure function in the  $r$ -evolution are given at the vertical (double) line through  $r_0$ . The  $y$ -evolution starts from the hadron impact factor given on the horizontal (dashed double) line through  $y_0$ .

The special features of the  $y$ -evolution show up in several characteristics of the hadronic final state. The inclusive jet distributions have been mentioned above. The transverse energy flow in the rapidity region away from the current jet has been observed [6] to be more pronounced at small  $x$ . It turns out that these data are not well described by particle production according to the mechanism of the GLAP equation. A good description at small  $x$  is achieved when using a production mechanism based on the BFKL equation [9].

The critical line, beyond which recombination is important, is estimated by using the simple unitarity argument [17] that the  $\gamma^*p$  cross section must be bounded by the geometric cross section. This should still be true if  $\gamma^*$  would interact strongly, i.e. if  $\alpha_{QED}$  is replaced by  $\alpha_S(Q^2)$ .

$$\frac{\alpha_S(Q^2)}{\alpha_{QED}} \sigma^{\gamma^*p} = \frac{\alpha_S(Q^2)}{Q^2} F_2(x, Q^2) < \pi R_p^2. \quad (3.9)$$

Substituting the result of the BFKL equation we obtain the limit of applicability of this equation. Unitarity demands to go beyond the leading logarithmic approximation. A minimal way to improve the unitarity properties of this approximation will be discussed in the last section.

An approach to unitarity improvement has been proposed by Gribov, Levin and Ryskin [17]. It amounts in including a quadratic term in the evolution equation. Such a term accounts for the splitting of the gluon ladder into two ladders, assuming a simple splitting vertex. The GLR equation has been studied in detail [18].

In QCD there are contributions with more than two reggeized gluons in the  $t$ -channel. In general they do not come in colour-singlet pairs. A recent study [19] of the perturbative QCD asymptotics of the  $\gamma^*p$  high mass diffraction shows clearly that the transition vertex of a two-reggeon state to a four-reggeon state in the  $t$ -channel is more complicated than the one assumed in the GLR approach.

## 4 The perturbative Regge asymptotics

In the kinematics Eq. (1) the hard scattering is determined by a small coupling. In the  $y$ -evolution ( $x_0 \rightarrow x \ll x_0, Q_0^2 \rightarrow Q^2 \sim Q_0^2$ ) the range of the transverse momentum integral is localized around  $Q_0$ . The diffusion in  $r$  is a higher order effect. Therefore one can start calculating with a fixed coupling (at scale  $Q_0^2$ ). The effects of the renormalization group are to be included in a next step [20].

We shall discuss some basic ideas of the perturbative Regge asymptotics. It is convenient to work with partial waves obtained by Mellin transformation of the  $x$ -dependence. The Mellin variable  $\omega$  is related to the angular momentum  $j$  by  $j = 1 + \omega$ .

The exchanged gluons become reggeons. The trajectory is  $1 + N\alpha_G(\kappa)$ , where in the leading logarithmic approximation we have

$$\alpha_G(\kappa) = \frac{g^2}{2(2\pi)^3} \int \frac{d^2\kappa' |\kappa^2|}{|\kappa'^2| |\kappa - \kappa'|^2}. \quad (4.10)$$

The asymptotics of the vacuum exchange channel is dominated by the exchange of an even number of reggeized gluons interacting by the exchange of s-channel gluons, Fig. 4.

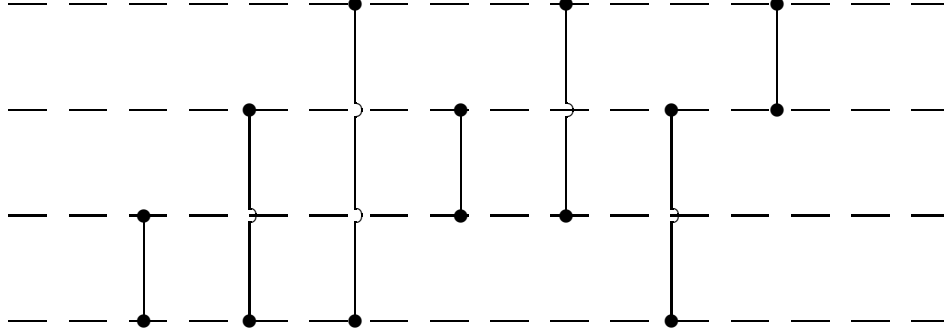


Fig. 4: Exchange of 4 reggeized gluons (dashed lines) interacting via s-channel gluons (full lines).

This interaction is determined by effective vertices obtained by adding to the appropriate projection of the usual three-gluon vertex the contributions of bremsstrahlung.

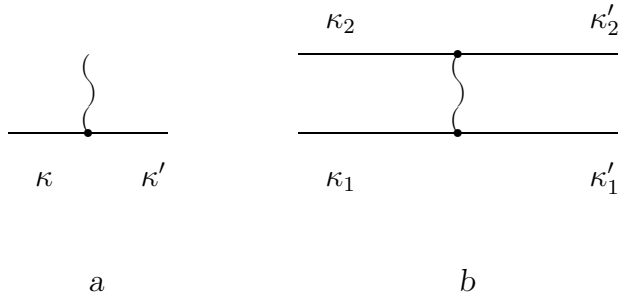


Fig. 5: a) Effective vertex of gluon emission from an exchanged (reggeized) gluon.  
 b) Interaction of two exchanged (reggeized) gluons.

For the two helicity states (represented by  $\phi$  and  $\phi^*$ ) of the s-channel gluon the leading



logarithmic approximation of the effective vertex is given by (compare Fig. 5a)

$$\frac{\kappa' \kappa^*}{(\kappa - \kappa')} \phi^* + \frac{\kappa^* \kappa'}{(\kappa^* - \kappa'^*)} \phi. \quad (4.11)$$

We represent the transverse momenta by complex numbers. Eq. (11) implies for the bare interaction kernel of two reggeized gluons (Fig. 5b)

$$\mathcal{H}^{(0)}(\kappa_1, \kappa_2; \kappa'_1, \kappa'_2) = \delta(\kappa_1 + \kappa_2 - \kappa'_1 - \kappa'_2) \frac{\kappa_1 \kappa'_1{}^* \kappa_2^* \kappa'_2 + \text{c.c.}}{|\kappa_1 - \kappa'_1|^2}. \quad (4.12)$$

The effective vertices are known [3] [21] since long. The projection on helicity states and the use of the complex notation lead to an essential simplification. Introducing appropriate fields one can write down the multi-Regge effective action [22] involving these vertices as interaction terms. It allows to reproduce in the easiest way the leading contributions in the Regge limit of QCD. This effective action can be derived from the original QCD action by separating the momentum modes of the gluon and the quark fields according to the multi-Regge kinematics and integrating out some "heavy" modes. There are remarkable relations between interaction terms describing scattering and production and between pure gluonic terms and those involving quarks. The longitudinal and the transverse dimensions are clearly separated in this effective action.

The case of two reggeized gluons is just the leading  $\ln x$  approximation discussed above. In order to satisfy the conditions of unitarity in the  $s$ -channel and in all sub-energy channels one has to include the contributions of arbitrary numbers of reggeons. This leads us to the generalized leading logarithmic approximation: The gluons in all  $s$ -channel intermediate states should obey the multi-Regge kinematics Eq. (5) and the effective vertices and the trajectories are taken in the leading logarithmic approximation Eq. (10), Eq.(11). Beyond the leading logarithmic approximation we encounter other contributions which arise e.g. if two  $s$ -channel gluons do not have a large invariant mass. They have to be accounted for in later steps as corrections to the trajectories and the effective vertices [23].

The partial-wave equation describing the exchange of  $r$  reggeons is a straightforward generalization of the BFKL equation [25] [24] [26]. It is represented graphically in Fig. 6.

Both the inhomogeneous term corresponding to  $r$  non-interacting reggeons and the interaction term involve the angular momentum part of the  $r$ -reggeon propagator,  $[\omega - N \sum_\ell \alpha_G(\kappa_\ell)]^{-1}$ . Both the trajectories Eq. (10) in this factor as well as the bare reggeon interaction kernel Eq. (12) involve infrared divergencies. In the colour-singlet channel, which is the one of physical interest, these divergencies cancel. We multiply the equation by the inverse of this factor and include the terms involving  $\alpha_G(\kappa)$  into the kernel,

$$\mathcal{H}(\kappa_1, \kappa_2; \kappa'_1, \kappa'_2) = |\kappa_1|^{-2} |\kappa_2|^{-2} \mathcal{H}^{(0)}(\kappa_1, \kappa_2; \kappa'_1, \kappa'_2) - \delta(\kappa_1 - \kappa'_1) (\alpha_G(\kappa_1) + \alpha_G(\kappa_2)), \quad (4.13)$$

which defines now a finite operator.

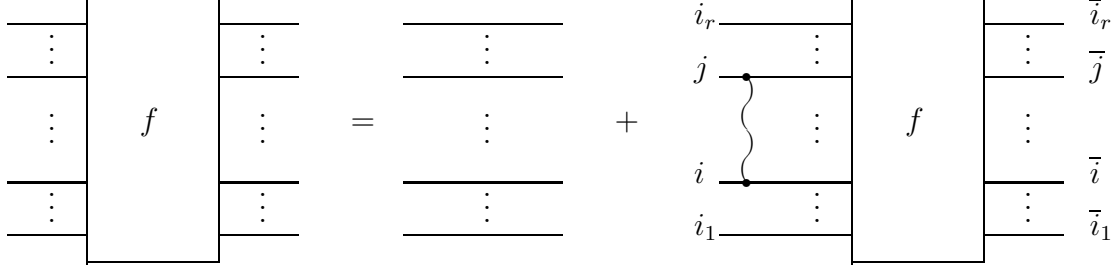


Fig. 6: Equation for the Green function of the  $r$ -reggeon exchange.

We change to the impact parameter representation by Fourier transformation with respect to the transverse momenta. The two-reggeon interaction is now represented by an operator acting on the transformed partial wave being now a function of the impact parameters  $x_\ell, \ell = 1, \dots, r$ . The operator is obtained from Eq. (13) by applying the substitutions

$$\begin{aligned} \kappa_1 &\rightarrow \partial_1^*, & \kappa_2^* &\rightarrow \partial_2, \\ |\kappa_1 - \kappa'_1|^{-2} &\rightarrow -\ln |x_{12}|^2, & \alpha_G(\kappa_1) &\rightarrow -\ln(\partial_1 \partial_1^*) + \psi(1). \end{aligned} \quad (4.14)$$

$\partial_1$  is the differentiation with respect to  $x_1$ ,  $-\psi(1)$  is the Euler - Mascheroni number and  $x_{12} = x_1 - x_2$ .

It is remarkable that the resulting operator decomposes into a holomorphic and an antiholomorphic part.

$$\begin{aligned} \hat{\mathcal{H}}_{GG} &= H_G + H_G^*, \\ H_G &= 2\psi(1) - \partial_1^{-1} \ln x_{12} \partial_1 - \partial_2^{-1} \ln x_{12} \partial_2 - \ln \partial_1 - \ln \partial_2. \end{aligned} \quad (4.15)$$

In the case of two reggeized gluons we obtain the homogeneous BFKL equation in the form

$$\omega f(\omega, x_1, x_2) = \frac{g^2 N}{8\pi^2} \hat{\mathcal{H}}_{GG} f(\omega, x_1, x_2). \quad (4.16)$$

Because of the decomposition Eq. (15) this equation allows a holomorphically factorized solution.

In the case  $r \geq 4$  the gauge group structure of the interaction prevents in general such a factorization. However in the large  $N$  approximation this factorization holds. Then the holomorphic part of the  $r$ -reggeon equation Fig. 6 corresponds just to the Schrödinger equation of a  $r$ -body system in one dimension with the interaction restricted to the nearest

neighbours and given by the unconventional hamiltonian  $-H_G$  in Eq. (15). We shall see that it is really an extraordinary hamiltonian.

We use operator relations expressing the derivative operator  $\partial$  as a similarity transformation of  $x^{-1}$ ,

$$\partial = \Gamma(x\partial + 1)^{-1} x^{-1} \Gamma(x\partial + 1) = \Gamma(-x\partial) x^{-1} \Gamma(-x\partial)^{-1}, \quad (4.17)$$

in order to obtain [27]

$$\ln \partial = \frac{1}{2} (\psi(x\partial) + \psi(1 - x\partial)) + (x\partial)^{-1} - \ln x. \quad (4.18)$$

The commutation relations between  $x$  and  $\partial$  imply

$$\partial^{-1} \ln x \partial = (x\partial)^{-1} \ln x (x\partial) = \ln x - (x\partial)^{-1}. \quad (4.19)$$

Using Eqs. (18) and (19) we obtain

$$H_G = 2\psi(1) - \frac{1}{2} (\psi(x_{12}\partial_1) + \psi(-x_{12}\partial_2) + \psi(1 - x_{12}\partial_1) + \psi(1 + x_{12}\partial_2)). \quad (4.20)$$

$\psi(z)$  is the digamma function,

$$\psi(1) - \psi(z) = \sum_{\ell=0}^{\infty} \left( \frac{1}{\ell + z} - \frac{1}{\ell + 1} \right). \quad (4.21)$$

We neglect the operator  $x_{12}(\partial_1 + \partial_2)$  in the following transformation. This is justified in our kinematics, because  $i(\partial_1 + \partial_2)$  is the momentum transfer operator and  $x_{12}$  is Fourier conjugated to  $\kappa$ . Then Eqs.(20) and (21) imply [27]

$$H_G = \chi_0(x_{12}^2 \partial_1 \partial_2), \quad \chi_0(z) = \sum_{\ell=0}^{\infty} \left( \frac{2\ell + 1}{\ell(\ell + 1) + z} - \frac{2}{\ell + 1} \right). \quad (4.22)$$

We have obtained  $H_G$  as a function of the Casimir operator,  $x_{12}^2 \partial_1 \partial_2$ , of holomorphic linear conformal transformations acting on functions of  $x_1$  and  $x_2$ . There is another way to show that  $H_G$  is conformally invariant. Therefore the result Eq. (22) holds in general irrespective of the approximation done in the last step.

The eigenvalues of the Casimir operator in the unitary representation are  $m(1 - m)$ , with  $m = \frac{1}{2} + i\nu + \frac{n}{2}$ .  $n$  runs over all integers and  $\nu$  runs over the real axis. In this way we obtain the eigenvalues of  $H_G$ . In particular we reproduce the eigenvalues of the BFKL equation Eq. (15) and the leading behaviour Eq. (2).

Including fermion exchange leads to an asymptotics down by powers of  $s$  compared to the exchange of gluons only. However in some channels the quantum numbers exclude the latter contribution. For example, fermion exchange contributions determine the small  $x$  behaviour of flavour non-singlet structure functions. The interaction between two reggeized fermions and between a reggeized fermion and a reggeized gluon are described in analogy to the gluon-gluon case discussed here. There is a peculiarity in the two-fermion

case because of double-logarithmic contributions [28]. The properties of holomorphic factorization and conformal symmetry hold for all two-reggeon interactions in QCD [29].

The holomorphic linear conformal transformations of the functions of  $x_1$  and  $x_2$  are generated by  $M_{12}^a = M_1^a + M_2^a$ , with  $M_1^+ = x_1^2 \partial_1$ ,  $M_1^- = \partial_1$ ,  $M_1^{(0)} = x_1 \partial_1$ . The commutation relations are the ones of the angular momenta modified by a sign. We introduce the  $2 \times 2$  matrix  $L_\ell(\theta)$ , one for each reggeon,  $\ell = 1, \dots, r$ ,

$$L_\ell(\theta) = \begin{pmatrix} \theta + M_\ell^{(0)} & M_\ell^- \\ -M_\ell^+ & \theta - M_\ell^{(0)} \end{pmatrix}. \quad (4.23)$$

The commutation relations of the generators  $M_\ell^a$  can be written in the  $R$ -matrix form [31],

$$(L_\ell(\theta_1) \otimes L_\ell(\theta_2)) R(\theta_1 - \theta_2) = R(\theta_1 - \theta_2) (L_\ell(\theta_2) \otimes L_\ell(\theta_1)). \quad (4.24)$$

Here  $\otimes$  denotes the tensor product.  $R(\theta)$  is the  $4 \times 4$  matrix the non-vanishing entries of which are  $R_{11,11}(\theta) = R_{22,22}(\theta) = \theta + 1$ ,  $R_{12,12}(\theta) = R_{21,21}(\theta) = 1$ ,  $R_{12,21}(\theta) = R_{21,12}(\theta) = \theta$ . It obeys the Yang-Baxter relation [30]. Therefore the operators  $\text{tr} \prod_{\ell=1}^r L_\ell(\theta)$  with  $\theta = \theta_1$  and  $\theta = \theta_2$  commute and generate by decomposition in  $\theta$   $(r-1)$  mutually commuting operators.

We see that the problem of the  $r$ -reggeon exchange in the large  $N$  limit leads to a periodic lattice spin model [31], a generalization of the Heisenberg XXX model. The spin  $\frac{1}{2}$  at each site in the Heisenberg model is here replaced by a representation of the group  $\text{SL}(2, \mathbb{R})$ . This lattice model is completely integrable [31] [32].

There is another  $R$ -matrix being an operator acting on functions of  $x_1$  and  $x_2$  and obeying

$$(L_1(\theta_1) \cdot L_2(\theta_2)) \mathbf{R}(\theta_2 - \theta_1) = \mathbf{R}(\theta_2 - \theta_1) (L_2(\theta_2) \cdot L_1(\theta_1)). \quad (4.25)$$

The dot denotes usual matrix multiplication.  $\mathbf{R}(\theta)$  generates another set of commuting operators. They commute also with the ones of the first set [32] [30]. The first non-trivial term in the  $\theta$ -decomposition contains just the hamiltonian  $-H_G$ .

$$\mathbf{R}(\theta) = P_{12} \left( 1 - \theta H_G + \mathcal{O}(\theta^2) \right). \quad (4.26)$$

With these results the problem of the  $r$ -reggeon exchange asymptotics is in principle solved in the large  $N$  approximation. It remains to work out the Improvements for  $N = 3$  and the phenomenological implication concerning the saturation of the increase of the structure functions at very small  $x$ .

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